# Winter Lab Meeting – Fall 20/21

## Kronecker Lemma and its Application

## Statment: Kronecker Lemma

If sequences $a_n \uparrow \infty$ and $\{\sum_{n=1}^{m} x_n/a_n\}_{m=1}^{\infty}$ converges, $\lim_{n\to\infty} a_n^{-1}\sum_{m=1}^{n} x_m \to 0$.

Take $a_0 = b_0 = 0$ and for $m \geq 1$, $b_m := \sum_{k=1}^{m} x_k/a_k$. Then $x_m = a_m(b_m - b_{m-1})$, so

$$a_n^{-1}\sum_{m=1}^{n} x_m = a_n^{-1}\left\{a_n b_n + \sum_{m=2}^{n} a_m b_m - \sum_{m=2}^{n} a_m b_{m-1}\right\} = b_n - \sum_{m=1}^{n}\frac{a_m - a_{m-1}}{a_n} b_{m-1}$$

It suffices to show $\lim_{m\to\infty}\sum_{m=1}^{n}\frac{a_m - a_{m-1}}{a_n} b_{m-1} = \lim_{n\to\infty} b_n =: b_\infty$. Take $B := \sup|b_n|$. For $\epsilon > 0$, as $\lim_{m\to\infty} b_m = b_\infty$, $\exists M \in \mathbb{N}$ s.t. $|b_m - b_\infty|$, $\forall m \geq M$. As $a_n \uparrow \infty$, $\exists N > M$ s.t. $a_n > 4Ba_m/\epsilon$, $\forall n \geq N$ (Equivalently, $a_M/a_n < \epsilon/(4B)$).

Noticing that $\sum_{m=1}^{k}(a_m - a_{m-1}) = a_k - a_0 = a_k$ holds for all $k \in \mathbb{N}$. Then whenever $n \geq N$,

$$\left|\sum_{m=1}^{n}\frac{a_m - a_{m-1}}{a_n} b_{m-1} - b_\infty\right| \overset{a_{m+1}\geq a_m}{\leq} \sum_{m=1}^{n}\frac{a_m - a_{m-1}}{a_n}|b_{m-1} - b_\infty|$$

$$= \sum_{m=1}^{M}\frac{a_m - a_{m-1}}{a_n}|b_{m-1} - b_\infty| + \sum_{m=M+1}^{n}\frac{a_m - a_{m-1}}{a_n}|b_{m-1} - b_\infty|$$

$$= \sum_{m=1}^{M}\frac{a_m - a_{m-1}}{a_n}(2B) + \sum_{m=1}^{M}\frac{a_m - a_{m-1}}{a_n}\left(\frac{\epsilon}{2}\right) < \left(\frac{\epsilon}{4B}\right)(2B) + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow \lim_{n\to\infty} a_n^{-1}\sum_{m=1}^{n} x_m = \lim_{n\to\infty} b_n - \lim_{n\to\infty}\sum_{m=1}^{n}\frac{a_m - a_{m-1}}{a_n} b_{m-1} = b_\infty - b_\infty = 0.$$

# Generalization of Kronecker Lemma

## Kronecker Lemma for Double Series

Let $\{a_{ij}\}_{i,j\in\mathbb{N}+}$ be a double sequence of positive number s.t.

- $a_{i+1,j} - a_{ij} \geq 0$ and $a_{i,j+1} - a_{ij} \geq 0$ for all $i, j$.
- $a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{ij}$ is of constant sign.
- $a_{ij} \to \infty$ as $\min(i,j) \to \infty$.

If $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}/a_{ij}$ converges, then $a_{mn}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \to 0$ as $\min(i,j) \to \infty$.

*Remark:*

1. Potentially useful to handling rectangular partial sums in research problems related to spatial setting.
2. In the above statement, the convergence of $\{a_{ij}\}_{i,j\in\mathbb{N}+}$ is defined as following: $\{a_{ij}\}_{i,j\in\mathbb{N}+}$ converges to $a$ iff $\exists a$ s.t. $\forall\epsilon > 0,\ \exists N \in \mathbb{N}$ s.t. whenever $\min(i,j) \geq N,\ |a_{ij} - a| < \epsilon$.
3. Refer to The Kronecker lemmas for multiple series and some applications for further generalizations

# Applications of Kronecker Lemma

1. Establish SLLN for Martingales.

**Kolmogorov's Variance Criterion**

Let $X_1, X_2, \cdots$ be independent, then (1) $\mathbb{E}(X_n) = 0$ for all $n$ and (2) $\sum_n \mathbb{E}(X_n^2) < \infty$ together implies $\sum_n X_n$ converges a.s.

2. Take $a_n = n^c$ in Kronecker Lemma for some $c > 0$. By Variance Criteria, $\mathbb{E}(X_n) = 0$ for all $n$ and $\sum_n \mathbb{E}(n^{-2c} X_n^2) < \infty$ implies $\sum_n n^{-c} X_n$ converges a.s. By Kronecker Lemma, it further implies that $n^{-c} \sum_{k=1}^n X_k \to 0$ a.s.
*Remark:* Tune the non-decreasing sequence $a_n$ to obtain desired constraint.