1 UI Martingales

1.1 Main Theorem

Let M be U.I. martingale, i.e. M is a Martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_n\}, \mathbb{P})$ and $\{M_n\}$ is U.I. family. Then

1. M_∞ exists a.s. in $\mathscr{L}^1$.
2. $M_n = \mathbb{E}(M_\infty|\mathscr{F}_n)$.

Interpretation: The best prediction or knowledge we have concerning the convergence limit M_∞ at time n is given by M_n

1.2 Recall of Previous Theorem

1. **[Section 13.2]** U.I. implies boundedness in $\mathscr{L}^1$ (Converse does not holds).
2. **[Section 11.5 Doob's Convergence Theeorem]**
   Let X be a supermartingale bounded in $\mathscr{L}^1$, i.e. $\sup_n \mathbb{E}|X_n| < \infty$. Then, almost surely, $X_\infty \triangleq \lim_{n\to\infty} X_N$ exists and is finite.
3. **[Section 13.7 A necessary and sufficient condition for $\mathscr{L}^1$ convergence]**
   Let $\{X_n\}$ be a sequence in $\mathscr{L}^1$, and let $X \in \mathscr{L}^1$. Then $X_n \to X$ in $\mathscr{L}^1$ if and only if both of the following two conditions are satisfied
   (a) $X_n \overset{pr}{\to} X$.
   (b) The sequence $\{X_n\}$ is UI.

2 Levy's 'Upward' Theorem

2.1 Main Theorem

Let $\xi \in \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ and $M_n \triangleq \mathbb{E}(\xi|\mathscr{F}_n)$ a.s. Then

1. M is U.I. Martingale.
2. $M_n \to \eta \triangleq \mathbb{E}(\xi|\mathscr{F}_\infty)$ a.s. in $\mathscr{L}^1$.

2.2 Recall of Previous Theorem

1. **[Section 13.4]** Let $X \in \mathscr{L}^1$. Then the class

$$\left\{\mathbb{E}(X|\mathscr{G}) : \mathscr{G} \text{ is a sub-}\sigma\text{-algebra of } \mathscr{G}\right\}$$

   is uniformly integrable.
2. **[Section 1.6b]** If two probability measures agree on a π-system, then they agree on the σ-algebra generated by that π-system.

3 Martingale Proof of Kolmogorov's 0-1 Law

3.1 Main Theorem

Let $X_1, X_2, \cdots$ be a sequence of independents R.V.s. Define $\mathcal{I}_n = \sigma(X_{n+1}, X_{n+2}, \cdots)$ and $\mathcal{I} = \cap_n I_n$ as the tail σ-algebra. Then

$$F \in \mathcal{I} \Rightarrow \mathbb{P}(F) \in \{0, 1\}.$$

3.2 Recall of Previous Theorem

[Section 4.11 Kolmogorov's 0-1 Law (Partial)] Let $X_1, \cdots, X_n$ be independent R.V.s. Define $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ and $\mathcal{I}_n = \sigma(X_{n+1}, X_{n+1}, \cdots)$. Then $\mathscr{F}_n$ and $\mathcal{I}_n$ are independent.

4 Levy's 'Downward' Theorem

4.1 Main Theorem

Consider the measure space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\left\{\mathscr{G}_{-n} : \ n \in \mathbb{N}\right\}$ be collection of sub σ-algebra of $\mathscr{F}$ s.t.

$$\mathscr{G}_{-\infty} \triangleq \cap_k \mathscr{G}_{-k} \subseteq \cdots \subseteq \mathscr{G}_{-(n+1)} \subseteq \mathscr{G}_{-n} \subseteq \cdots \subseteq \mathscr{G}_{-1}$$

Let $\gamma \in \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ and define $M_{-n} \triangleq \mathbb{E}(\gamma|\mathscr{G}_{-n})$. Then

1. $M_{-\infty} \triangleq \lim_{n\to\infty} M_{-n}$ exists a,s. in $\mathscr{L}^1$.
2. $M_\infty = \mathbb{E}(\gamma|\mathscr{G}_{-\infty})$ a.s.

4.2 Recall of Previous Theorem

[Section 13.6 Bounded Convergence Theorem] Let $\{X_n\}$ be a sequence of R.V.s, and let X be a R.V. Suppose $X_n \overset{pr}{\to} X$ and that for some $K \in [0, \infty)$, we have for every n and ω,

$$|X_n(\omega)| \leq K$$

Then $\mathbb{E}(|X_n - X|) \to 0$

5 Martingale Proof of Strong Law of Large Number

Let $X_1, X_2, \cdots$ be IID R.V.s with $\mu \triangleq \mathbb{E}|X_k| < \infty, \ \forall k \in \mathbb{N}^+$. Define $S_n \triangleq X_1 + \cdots + X_n$, then

$$n^{-1}S_n \to \mu \text{ a.s. in } \mathscr{L}^1.$$

Remark: We did provide proof of SLLN before but only almost sure convergence is shown, where the Martingale proof also shows the $\mathscr{L}^1$ convergence.

6 Doob's Submartingale Inequality

Let Z be non-negative submartingale. Then for $c > 0$,

$$c\mathbb{P}\left(\sup_{k\leq n} Z_k \geq c\right) \leq \mathbb{E}\left\{Z_n; \sup_{k\leq n} Z_k \geq c\right\} \leq \mathbb{E}(Z_n)$$

6.1 Trivial Lemma

If M is a Martingale, c is a convex function, and $\mathbb{E}|c(M_n)| < \infty, \ \forall n$, then $c(M)$ os a Submartingale.

6.2 Application: Kolmogorov's inequality

Let $(X_N : n \in \mathbb{N})$ be a sequence of independent zero-mean R.V.s in $\mathscr{L}^2$. Define $\sigma_k^2 \triangleq \mathbb{V}ar X_k$.

$$S_n \triangleq X_1 + \cdots + X_n, \quad V_n \triangleq \mathbb{V}ar(S_n) = \sum_{k=1}^n \sigma_k^2.$$

Then for $c > 0$,

$$c^2\mathbb{P}\left(\sup_{k\leq n} |S_k| \geq c\right) \leq V_n$$

7 Law of Iterated Logarithm: special case

Let $X_1, \cdots, X_n \overset{iid}{\sim} N(0,1)$. Define $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\limsup_{n\to\infty} \frac{S_n}{(2n \log\log n)^{1/2}} = 1.$$

8 A standard estimate on the normal distribution

Suppose $X \sim N(0,1)$, then $\forall x \in \mathbb{R}$, $\mathbb{P}(X > x) = 1 - \Phi(x) = \int_x^\infty \varphi(y)dy$, where $\varphi(y) = (2\pi)^{-1/2}e^{-y^2/2}$ is the density of $N(0,1)$. Then for $x > 0$,

1. $\mathbb{P}(X > x) \le x^{-1}\varphi(x)$.
2. $\mathbb{P}(X > x) \ge (x + x^{-1})^{-1}\varphi(x)$.

9 Remarks on exponential bounds; large-deviation theory

Omitted

10 A consequence of Holder's inequality

Suppose X and Y are non-negative R.V.s such that

$$c\mathbb{P}(X \ge c) \le \mathbb{E}(Y; X \ge c),\ \forall c > 0$$

Then, for $p > 1$ and $p^{-1} + q^{-1} = 1$, we have

$$\|X\|_p \le q\|Y\|_p$$

11 Doob's $\mathcal{L}^p$ inequality

1. Let $p > 1$ and define q so that $p^{-1} + q^{-1} = 1$. Let Z be non-negative submartingale bounded in $\mathcal{L}^p$. Define $Z^* \triangleq \sup_{k\in\mathbb{Z}^+} Z_k$. Then $Z^* \in \mathcal{L}^p$ and indeed

$$\|Z^*\|_p \le q \sup_r \|Z_r\|_p$$

, i.e. the submartingale is therefore dominated by the element Z^* in $\mathcal{L}^p$. Also, as $n \to \infty$, $Z_\infty \triangleq \lim_{n\to\infty} Z_n$ exists a.s. in $\mathcal{L}^p$ and

$$\|Z_\infty\|_p = \sup_r \|Z_r\|_p =\uparrow \lim_{r\to\infty} \|Z_r\|_p.$$

2. If Z is of the form $|M|$, where M is martingale bounded in $\mathcal{L}^p$, then $M_\infty = \lim_{n\to\infty} M_n$ exits a.s. and in $\mathcal{L}^p$. And of course $Z_\infty = |M_\infty|$ a.s.

12 Kakutani's Theorem on 'Product' Martingales

Let $X_1, X_2, \cdots$ be independent non-negative R.V.s, each of mean 1. Define $M_0 \triangleq 1$, and for $n \in \mathbb{N}$, let

$$M_n \triangleq X_1 X_2 \cdots X_n.$$

Then M is a non-negative martingale, so that

$$M_\infty \triangleq \lim_{n\to\infty} M_n \quad \text{exists a.s.}$$

And the following five statements are equivalent.

1. $\mathbb{E}(M_\infty) = 1$.
2. $M_n \to M_\infty$ in $\mathscr{L}^1$.
3. M is U.I.
4. $\prod a_n > 0$, where $0 < a_n \triangleq \mathbb{E}(X_n^{1/2}) \leq 1$.
5. $\sum(1 - a_n) < \infty$.

If one of the above statements fails to hold, then $\mathbb{P}(M_\infty = 0) = 1$.

13 Appendix: $\mathscr{F}_T$ and OST for U.I. Martingale

13.1 Definition of $\mathscr{F}_T$

Let T be a stopping time. Then, for $F \subseteq \Omega$, we say that $F \in \mathscr{F}_T$ if

$$F \cap \{T = n\} \in \mathscr{F}_n, \ \forall n \in \mathbb{Z}^+ \cup \{\infty\}$$

Then $\mathscr{F}_T$ is also a σ-algebra.

Let X be a supermartingale. Let T be a stopping time such that, for some $N \in \mathbb{N}$, $T(\omega) \leq N, \ \forall \omega$. Then $X_T \in \mathscr{L}^1(\Omega, \mathscr{F}_T, \mathbb{P})$ and

$$\mathbb{E}(X_N | \mathscr{F}_T) \leq X_T$$

Let M be a UI Martingale. Then, for any stopping time T, $\mathbb{E}(M_\infty | \mathscr{F}_T) = M_T$

13.2 Doob's OST for U.I. Martingales

If M is a UI Martingale and S and T are both stopping time with $S \leq T$, then

$$\mathbb{E}(M_T | \mathscr{F}_S) = M_S, \ \text{a.s.}$$

If M is a UI Martingale, and T is a stopping time, then $\mathbb{E}(|M_T|) < \infty$ and

$$\mathbb{E}(M_T) = \mathbb{E}(M_0)$$