Probability with Martingales

Chapter 14: UI Martingales (Research Group Reading)

Agenda

Uniformly Integrable Martingale

Theorem

Let M be U.I. martingale, i.e. M is a Martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_n\}, \mathbb{P})$ and $\{M_n\}$ is U.I. family. Then

1. M_∞ exists a.s. in $\mathscr{L}^1$.
2. $M_n = \mathbb{E}(M_\infty|\mathscr{F}_n)$.

Proof of 1

M is U.I. implies that M is Martingale bounded in $\mathscr{L}^1$ and hence by Doob's 'Forward' Convergence Theorem, $M_\infty \triangleq \lim_{n\to\infty} M_n$ exist a.s., i.e. $M_n \overset{a.s.}{\to} M_\infty$ and it implies that $M_n \overset{pr}{\to} M_\infty$. By Section 13.7, $\mathscr{L}^1$ convergence holds, i.e.

$$\mathbb{E}|M_n - M_\infty| \to 0.$$

Proof of 2

We proceed step by step.

1. $\forall F \in \mathscr{F}_n, \forall r \geq n, \mathbb{E}(M_r; F) = \mathbb{E}(M_n; F)$.
   Trivial for $r = n$. Consider the case $r > n$. Noticing that $\mathscr{F}_n \subseteq \mathscr{F}_{r-1}$, then by property of conditional expectation, we have
   $$\mathbb{E}(M_r|\mathscr{F}_n) = \mathbb{E}\Big\{\mathbb{E}(M_r|\mathscr{F}_{r-1})|\mathscr{F}_n\Big\} = \mathbb{E}(M_{r-1}|\mathscr{F}_n) = \cdots = \mathbb{E}(M_n|\mathscr{F}_n)$$
   Then for $F \in \mathscr{F}_n, \mathbb{E}(M_r; F) = \mathbb{E}(M_n; F)$.
2. $|\mathbb{E}(M_r; F) - \mathbb{E}(M_\infty; F)| = |\mathbb{E}(M_r - M_\infty; F)| \overset{\text{(Jensen)}}{\leq} \mathbb{E}(|M_r - M_\infty|; F) \overset{\text{(Positivity)}}{\leq} \mathbb{E}|M_r - M_\infty|$. It follows by the $\mathscr{L}^1$ convergence that
   $$\lim_{r\to\infty} |\mathbb{E}(M_r; F) - \mathbb{E}(M_\infty; F)| \leq \lim_{r\to\infty} \mathbb{E}|M_r - M_\infty| = 0$$
   and hence $\lim_{r\to\infty} \mathbb{E}(M_r; F) = \mathbb{E}(M_\infty; F)$
3. By taking limit on equality obtained in (1) w.r.t to r, we obtained
   $$\mathbb{E}(M_\infty; F) = \mathbb{E}(M_n; F) \text{ a.s.}$$

Agenda

Levy's Upward Theorem

Theorem

Let $\xi \in \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ and $M_n \triangleq \mathbb{E}(\xi|\mathscr{F}_n)$ a.s. Then

1. M is U.I. Martingale.
2. $M_n \to \eta \triangleq \mathbb{E}(\xi|\mathscr{F}_\infty)$ a.s. in $\mathscr{L}^1$.

Proof: [M is U.I. Martingale]

1. Measurability: Follows from definition of conditional expectation.
2. Integrability: As $\xi \in \mathscr{L}^1$,
$$\mathbb{E}|M_n| = \mathbb{E}\Big\{|\mathbb{E}\big(\xi|\mathscr{F}_n\big)|\Big\} \leq \mathbb{E}\Big\{\mathbb{E}\Big(|\xi|\Big|\mathscr{F}_n\Big)\Big\} = \mathbb{E}|\xi| < \infty$$
3. Martingale Property:
$$\mathbb{E}(M_{n+1}|\mathscr{F}_n) = \mathbb{E}\Big\{\mathbb{E}(\xi|\mathscr{F}_{n+1})\Big|F_n\Big\} = \mathbb{E}(\xi|\mathscr{F}_n) = M_n$$

By section 13.4, for $\xi \in \mathscr{L}^1$, the class $\Big\{\mathbb{E}(\xi|\mathscr{F}_n) : n \in \mathbb{Z}\Big\}$ is U.I. and hence M is U.I. Martingale

Proof: [$M_n \to \eta \triangleq \mathbb{E}(\xi|\mathscr{F}_\infty)$ a.s. in $\mathscr{L}^1$]

As we can write $\xi = \xi^+ - \xi^-$ for some non-negative R.V. ξ^+, ξ^-, WLOG, assume $\xi \geq 0$. Define two measure $\mathbb{Q}_1, \mathbb{Q}_2$ on the measurable space $(\Omega, \mathscr{F}_\infty)$ by

$$\mathbb{Q}_1(F) = \mathbb{E}(\eta; F) \quad \text{and} \quad \mathbb{Q}_2(F) = \mathbb{E}(M_\infty; F), \text{ for } F \in \mathscr{F}_\infty.$$

As $\mathbb{E}(\eta|\mathscr{F}_n) = \mathbb{E}\Big\{\mathbb{E}(\xi|\mathscr{F}_\infty)|\mathscr{F}_n\Big\} = \mathbb{E}(\xi|\mathscr{F}_n) = M_n = \mathbb{E}(M_n|\mathscr{F}_n)$. Then for $F \in \mathscr{F}_n$,

$$\mathbb{Q}_1(F) = \mathbb{E}(\eta; F) = \mathbb{E}(M_n; F) \overset{(14.1)}{=} \mathbb{E}(M_\infty; F) = Q_2(F).$$

, i.e. $\mathbb{Q}_1, \mathbb{Q}_2$ agrees on the π-system $\cup_n \mathscr{F}_n$ and hence by Lemma 1.6, i.e. two probability measure agree on a π system implying the agreement on the σ algebra generated by that π-system. In our case, $\mathbb{Q}_1, \mathbb{Q}_2$ agree on $\mathscr{F}_\infty \triangleq \sigma(\cup_n \mathscr{F}_n)$.

As both η, M_∞ are $\mathscr{F}_\infty$-measurable. Or formally, M_∞ has to be $\mathscr{F}_\infty$-measurable by defining $M_\infty \triangleq \limsup M_n$. Hence the event $F \triangleq \{\omega : \eta > M_\infty\} \in \mathscr{F}_\infty$. As $\mathbb{Q}_1(F) = \mathbb{Q}_2(F)$,

$$\mathbb{E}(\eta - M_\infty; F) = \mathbb{E}(\eta; F) - \mathbb{E}(M_\infty; F) = 0 \Rightarrow \mathbb{P}(M_\infty > \eta) = 0$$

By symmetry, $\mathbb{P}(M_\infty < \eta) = 0$ and hence $\mathbb{P}(M_\infty = \eta) = 1$

Technical Remark

In the previous proof, we has to apply Lemma 1.6 to justify the agreement of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ on $\mathscr{F}_\infty$. The key is that in general, for increasing sub-σ-algebra of $\mathscr{F}$, i.e.

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_n \subseteq \cdots$$

, $\cup_n \mathscr{F}_n$ is **NOT** itself a σ-algebra but only π-system and thus the extension lemma is required. We then give an explicit example.

Example 1

Consider the measure space $(\Omega, \mathscr{F}, \{\mathscr{F}_n : n \in \mathbb{N}\}, \mathbb{P})$, where $\Omega = [0,1], \mathscr{F} = \mathscr{B}([0,1]), \mathscr{F}_k = \sigma\left\{[0, 2^{-k}], [2^{-k}, 2 \times 2^{-k}], \cdots, [(2^k - 1)2^{-k}, 1]\right\}$ and $\mathbb{P}$ denotes the Lebesgue Measure on $[0,1]$. Suppose $\cup_i \mathscr{F}_i$ is a σ-algebra.
Pick arbitrary $x \in [0,1] \backslash \mathbb{Q}$ and construct an sequence $s_1, s_2, \cdots$ converging to x **from the left**, where $s_i = (2^k - 1)/2^k$ for some $k \in \mathbb{N}$, its existence is guaranteed by the binary representation. Therefore,

$$[x, 1] = \cap_{i \in \mathbb{N}} [s_i, 1] \in \cup_{i \in \mathbb{N}} \mathscr{F}_i$$

However, for each fixed k, the elements in $\mathscr{F}_k$ are either rational number or intervals with rational number as its endpoints, i.e. $[x, 1] \notin \mathscr{F}_i$ for all $i \in \mathbb{N}$. Contradiction arises and hence $\cup_{i \in \mathbb{N}} \mathscr{F}_i$ is NOT a σ-algebra.

Agenda

Martingale Proof of Kolmogorov's 0-1 Law

Theorem

Let $X_1, X_2, \cdots$ be a sequence of independents R.V.s. Define $\mathcal{I}_n = \sigma(X_{n+1}, X_{n+2}, \cdots)$ and $\mathcal{I} = \cap_n I_n$ as the tail σ-algebra. Then

$$F \in \mathcal{I} \Rightarrow \mathbb{P}(F) \in \{0, 1\}.$$

Proof

Define $\mathscr{F}_n \triangleq \sigma(X_1, \cdots, X_n)$. Let $F \in \mathcal{I}$ and $\eta \triangleq \mathbb{1}(F)$. Since $\eta \in b\mathscr{F}_\infty$, i.e. Bounded and $\mathscr{F}_\infty$-measurable function. By Levy's Upward Theorem,

$$\eta = \mathbb{E}(\eta|\mathscr{F}_\infty) = \lim_{n\to\infty} \mathbb{E}(\eta|\mathscr{F}_n) \text{ a.s.}$$

Also, as $\mathcal{I}_n \supset \mathcal{I}_{n+1} \supset \cdots \supset \mathcal{I}$, then η is $\mathcal{I}_n$-measurable and hence independent of $\mathscr{F}_n$ for each $n \in \mathbb{N}$. By the independence, we have

$$\{0, 1\} \ni \eta = \lim_{n\to\infty} \mathbb{E}(\eta|\mathscr{F}_n) = \lim_{n\to\infty} \mathbb{P}(F) = \mathbb{P}(F)$$

Agenda

Levy's Downward Theorem

Theorem

Consider the measure space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\left\{\mathscr{G}_{-n} : n \in \mathbb{N}\right\}$ be collection of sub σ-algebra of $\mathscr{F}$ s.t.

$$\mathscr{G}_{-\infty} \triangleq \cap_k \mathscr{G}_{-k} \subseteq \cdots \subseteq \mathscr{G}_{-(n+1)} \subseteq \mathscr{G}_{-n} \subseteq \cdots \subseteq \mathscr{G}_{-1}$$

Let $\gamma \in \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ and define $M_{-n} \triangleq \mathbb{E}(\gamma|\mathscr{G}_{-n})$. Then

1. $M_{-\infty} \triangleq \lim_{n\to\infty} M_{-n}$ exists a,s. in $\mathscr{L}^1$.
2. $M_{-\infty} = \mathbb{E}(\gamma|\mathscr{G}_{-\infty})$ a.s.

Proof

1. As $\mathbb{E}|M_{-n}| = \mathbb{E}\left\{|\mathbb{E}(\gamma|\mathscr{G}_{-n})|\right\} \leq \mathbb{E}\left\{\mathbb{E}(|\gamma| \big| \mathscr{G}_{-n})\right\} = \mathbb{E}|\gamma| < \infty$, we have $\sup_n \mathbb{E}|M_{-n}| \leq \mathbb{E}|\gamma| < \infty$. By Doob's Forward Convergence Theorem, $M_{-\infty} \triangleq \lim_{n\to\infty} M_{-n}$ exists a.s. in $\mathscr{L}^1$.
2. For $G \in \mathscr{G}_{-\infty} \subseteq \mathscr{G}_{-r}$, we have
$$\mathbb{E}(M_{-r}|\mathscr{G}_{-\infty}) = \mathbb{E}\left\{\mathbb{E}(\gamma|\mathscr{G}_{-r})|\mathscr{G}_{-\infty}\right\} = \mathbb{E}(\gamma|\mathscr{G}_{-\infty})$$
Notice that $\gamma \in \mathscr{L}^1$ implies $\mathbb{E}(\gamma|\mathscr{G}_{-\infty})$ being bounded. Therefore, by bounded convergence theorem, $\mathbb{E}(\gamma|G) = \mathbb{E}(M_{-\infty}|G) = M_{-\infty}$

Agenda

Martingale Proof of the Strong Law

Theorem

Let $X_1, X_2, \cdots$ be IID R.V.s with $\mathbb{E}|X_k| < \infty,\ \forall k \in \mathbb{N}^+$. Define $\mu \triangleq \mathbb{E}(X_1)$ and $S_n \triangleq X_1 + \cdots + X_n$, then $n^{-1}S_n \to \mu$ a.s. in $\mathscr{L}^1$.

Proof

Define $\mathscr{G}_{-n} \triangleq \sigma(S_n, S_{n+1}, \cdots)$ and $\mathscr{G}_{-\infty} \triangleq \cap_n \mathscr{G}_{-n}$. Recall from Section 9.11,

$$\mathbb{E}(X_1|\mathscr{G}_{-n}) = \mathbb{E}(X_1|S_n) = \cdots = E(X_n|S_n) = \mathbb{E}\left\{n^{-1}(X_1 + \cdots + X_n)|S_n\right\} = n^{-1}S_n$$

As $X_1 \in \mathscr{L}^1$ and $\mathscr{G}_{-1} \supseteq \mathscr{G}_{-2} \supseteq \cdots \supseteq \mathscr{G}_{-n} \supseteq \mathscr{G}_{-(n+1)} \supseteq \cdots \mathscr{G}_{\infty}$. By Levy's Downward Theorem,

$$\lim_{n\to\infty} n^{-1}S_n = \lim_{n\to\infty} \mathbb{E}(X_1|\mathscr{G}_{-n}) \triangleq L \text{ exists a.s. in } \mathscr{L}^1$$

As the limit exist and by the IID property, we can write for each $k \in \mathbb{N}$, $L = \limsup_{n\to\infty}(X_{k+1} + \cdots + X_{k+n})/n$ so that $L \in m\mathcal{I}_k$, where $\mathcal{I}_k \triangleq \sigma(X_{k+1}, X_{k+2}, \cdots)$. By Kolmogorov's 0-1 Law, as $L \in m\mathcal{I}_k$ for all k, $\mathbb{P}(L = c) = 1$ for some fixed $c \in \mathbb{R}$. By Fatou's (and its Reverse) lemma,

$$\mathbb{E}(L) = \mathbb{E}(\lim_{n\to\infty} n^{-1}S_n) \le \liminf_{n\to\infty} \mathbb{E}(n^{-1}S_n) = \liminf_{n\to\infty} \mu = \mu$$

$$\mathbb{E}(L) = \mathbb{E}(\lim_{n\to\infty} n^{-1}S_n) \ge \limsup_{n\to\infty} \mathbb{E}(n^{-1}S_n) = \limsup_{n\to\infty} \mu = \mu$$

Hence $c = \mathbb{E}(c) = \mathbb{E}(L) = \mu$ a.s.

Agenda

Doob's Submartingale Inequality

Theorem

Let Z be non-negative submartingale. Then for $c > 0$,

$$c\mathbb{P}\left(\sup_{k\le n} Z_k \ge c\right) \le \mathbb{E}\left\{Z_n; \sup_{k\le n} Z_k \ge c\right\} \le \mathbb{E}(Z_n)$$

Proof

Let $F \triangleq \left\{\sup_{k\le n} Z_k \ge c\right\}$. Then F can be written as a disjoint union $F = F_0 \cup F_1 \cup \cdots \cup F_n$, where $F_0 \triangleq \{Z_0 \ge c\}$ and

$$F_k \triangleq \{Z_0 < c\} \cap \{Z_1 < c\} \cap \cdots \cap \{Z_k \ge c\}$$

Obviously, $F_k \in \mathscr{F}_k$ and $Z_k \ge c$ on F_k. Hence for $n \ge k$,

$$\mathbb{E}(Z_n; F_k) \ge \mathbb{E}(Z_k; F_k) \ge c\mathbb{P}(F_k)$$

, where the first inequality sign follows from the submartingale property. Summing over the index k from 1 to n gives the desired inequality.

Trivial Lemma

Lemma

If M is a Martingale, c is a convex function, and $\mathbb{E}|c(M_n)| < \infty, \ \forall n$, then $c(M)$ is a Submartingale.

Proof:

Measruability and Integrability is obvious. The supermartingale condition follows from Jensen's inequality.

$$\mathbb{E}\Big\{c(M_{n+1})|\mathscr{F}_n\Big\} \geq c\Big(\mathbb{E}\{M_{n+1}|\mathscr{F}_n\}\Big) = c(M_n)$$

Theorem

Let $(X_N : n \in \mathbb{N})$ be a sequence of independent zero-mean R.V.s in $\mathscr{L}^2$. Define $\sigma_k^2 \triangleq \mathbb{V}ar X_k$.

$$S_n \triangleq X_1 + \cdots + X_n, \quad V_n \triangleq \mathbb{V}ar(S_n) = \sum_{k=1}^{n} \sigma_k^2.$$

Then for $c > 0$,

$$c^2 \mathbb{P}\left(\sup_{k \leq n} |S_k| \geq c \right) \leq V_n$$

Proof

As S is a Martingale, S^2 is a Submartingale. It follows from the Doob's Submartingale Inequality that for $c > 0$,

$$c^2 \mathbb{P}\left(\sup_{k \leq n} |S_n| \geq c \right) = c^2 \mathbb{P}\left(\sup_{k \leq n} S_n^2 \geq c^2 \right) \leq \mathbb{E}(S_n^2) = V_n$$

Agenda

Law of Iterated Logarithm

Theorem

Let $X_1, \cdots, X_n \overset{iid}{\sim} N(0,1)$. Define $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\limsup_{n\to\infty} \frac{S_n}{(2n\log\log n)^{1/2}} = 1.$$

Proof [1. Exponential Bound]

Define $\mathscr{F}_n \triangleq \sigma(X_1, \cdots, X_n)$. Recall section 10.4a, S is a martingale w.r.t. $\{\mathscr{F}_n\}$. As $S_n \sim N(0,n)$, then recall the MGF of normal density, we have $\mathbb{E}(e^{\theta S_n}) = e^{\theta^2 n/2} < \infty$. Notice that $x \mapsto e^{\theta x}$ is convex in $\mathbb{R}$ and hence $e^{\theta S_n}$ is a submartingale. By Doob's submartingale inequality, as $e^{\theta S_n}$ is non-negative, we have

$$e^{\theta c}\mathbb{P}\left(\sup_{k\le n} S_k \ge c\right) = e^{\theta c}\mathbb{P}\left(\sup_{k\le n} e^{\theta S_k} \ge e^{\theta c}\right) \le \mathbb{E}(e^{\theta S_n}) = e^{\theta^2 n/2}$$

and hence $\mathbb{P}\left(\sup_{k\le n} S_k \ge c\right) \le e^{\frac{1}{2}n\theta^2 - c\theta}$. As θ is arbitrary, we can find a more sensible bound by setting $\theta = c/n$ and so $\mathbb{P}\left(\sup_{k\le n} S_k \ge c\right) \le e^{-\frac{c^2}{2n}}$.
(Recall section 6.4: Selection of optimum bound)

Proof [2. Obtaining an upper bound]

Let $K \in (1, \infty)$. Define $h_(n) \triangleq (2n \log\log n)^{1/2}$ and $c_n \triangleq Kh(K^{n-1})$. Then

$$\mathbb{P}\left(\sup_{k \le K^n} S_k \ge c_n\right) \le e^{-\frac{1}{2}c_n^2/K^n} = \exp\left\{-\frac{1}{2K^n}K^2 2K^{n-1}\log\log(K^{n-1})\right\}$$
$$= \exp\left\{-K\log\left[(n-1)\log K\right]\right\} = (n-1)^{-K}(\log K)^{-K}.$$

Denote $A_n = \{\omega \in \Omega : \sup_{k \le K^n} S_k \ge c_n\}$. It follows that

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n) \le \mathbb{P}(A_1) + \sum_{n=2}^{\infty}(n-1)^{-K}(\log K)^{-K} = 1 + (\log K)^{-K}\sum_{n=1}^{\infty} n^{-K} < \infty$$

whenever $K > 1$. By Borel Cantelli 1, $\mathbb{P}(A_n$ i.o.$) = 0$ and hence $\mathbb{P}(A_n^c$ e.v.$) = 1$. Therefore, almost surely, $\exists n_0(\omega)$ s.t. $\forall n \ge n_0(\omega)$, we have for $k \in [K^{n-1}, K^n]$,

$$S_k \le \sup_{k \le K^n} S_k \le c_n = Kh(K^{n-1}) \le Kh(k)$$

where the last inequality follows from monotonicity of $h(\cdot)$. Hence,

$$\limsup_{n\to\infty} h(k)^{-1}S_k \le K \text{ a.s.}$$

As the only requirement above is that $K \in (1, \infty)$. We can take $K_t = 1 + t^{-1}$ to yield $\limsup_{n\to\infty} h(k)^{-1}S_k \le K_t$ a.s.. By taking limit as $t \to \infty$, we obtain

$$\limsup_{n\to\infty} h(k)^{-1}S_k \le 1 \text{ a.s.}$$

Law of Iterated Logarithm

Proof [3. Obtaining a lower bound]

Let $N \in \mathbb{N} \cap [2, \infty)$. Let $\epsilon \in (0, 1)$. For simplicity, denote $S(r) \triangleq S_r$. Define the event

$$F_n \triangleq \left\{ S(N^{n+1}) - S(N^n) > (1-\epsilon)h(N^{n+1} - N^n) \right\}.$$

Notice that $S(N^{n+1}) - S(N^n) \sim N(0, N^{n+1} - N^n)$. Therefore, $\{S(N^{n+1}) - S(N^n)\}/\sqrt{N^{n+1} - N^n} \sim N(0, 1)$. By result in section 14.8(b), we obtain

$$\mathbb{P}(F_n) = \mathbb{P}\left(\frac{S(N^{n+1}) - S(N^n)}{(N^{n+1} - N^n)^{1/2}} \geq (1-\epsilon)\frac{h(N^{n+1} - N^n)}{(N^{n+1} - N^n)^{1/2}} \right) \geq \frac{1}{\sqrt{2\pi}}(y + y^{-1})^{-1} e^{-\frac{y^2}{2}}$$

, where $y \triangleq (1-\epsilon)\sqrt{2 \log\log(N^{n+1} - N^n)}$. After tedious algebra, we obtain $\mathbb{P}(F_n) \approx (n \log N)^{-(1-\epsilon)^2}$ and hence $\sum_{n=1}^{\infty} \mathbb{P}(F_n) = \infty$. As $X_1, X_2, \cdots$ are IID, $\{F_n : n \in \mathbb{N}\}$ is sequence of independent events. By Borel-Cantelli Lemma 2, $\mathbb{P}(\limsup_{n\to\infty} F_n) = 1$, i.e. almost surely infinitely many F_n occur.

Proof [3. Obtaining a lower bound] Cont.

Explicitly, for infinitely many n, $S(N^{n+1}) > (1-\epsilon)h(N^{n+1} - N^n) + S(N^n)$. But by step 2, for large n, we have $S(N^n) \leq 2h(N^n)$. As $S(N^n) \overset{d}{=} -S(N^n)$, we have $S(N^n) \geq -2h(N^n)$. So for infinitely many n,

$$S(N^{n+1}) > (1-\epsilon)h(N^{n+1} - N^n) - 2h(N^n).$$

It follows from definition that

$$\begin{aligned}\limsup_{k\to\infty} h(k)^{-1}S_k \geq \limsup_{n\to\infty} h(N^{n+1})^{-1}S(N^{n+1}) &\geq (1-\epsilon)\frac{h(N^{n+1}-N^n)}{h(N^{n+1})} - 2\frac{h(N^n)}{N^{n+1}} \\ &\approx (1-\epsilon)\sqrt{\frac{N^{n+1}-N^n}{N^{n+1}}} - 2\sqrt{\frac{N^n}{N^n+1}} = (1-\epsilon)(1-N^{-1})^{1/2} - 2N^{-1/2}\end{aligned}$$

Letting $N \to \infty$ and $\epsilon \downarrow 0$, we have $\limsup_{k\to\infty} h(k)^{-1}S_k \geq 1$. Recall from part (2) that $\limsup_{k\to\infty} h(k)^{-1}S_k \leq 1$. Therefore,

$$\limsup_{k\to\infty} h(k)^{-1}S_k = 1$$

Agenda

A standard estimate on the normal distribution

Theorem

Suppose $X \sim N(0,1)$, then $\forall x \in \mathbb{R}$, $\mathbb{P}(X > x) = 1 - \Phi(x) = \int_x^\infty \varphi(y)dy$, where $\varphi(y) = (2\pi)^{-1/2}e^{-y^2/2}$ is the density of $N(0,1)$. Then for $x > 0$,

1. $\mathbb{P}(X > x) \le x^{-1}\varphi(x)$.
2. $\mathbb{P}(X > x) \ge (x + x^{-1})^{-1}\varphi(x)$.

Proof

1. Notice that $\varphi'(y) = -y\varphi(y)$. Therefore, for $x > 0$,
$$\varphi(x) = \int_{-\infty}^x \varphi(y)'dy = \int_x^\infty y\varphi(y)dy \ge x\int_x^\infty \varphi(y)dy = x\mathbb{P}(X > x)$$
It follows that $\mathbb{P}(X > x) \le x^{-1}\varphi(x)$.
2. Notice that $\left(y^{-1}\varphi(y)\right)' = -(1 + y^{-2})\varphi(y)$. Therefore, for $x > 0$,
$$x^{-1}\varphi(x) = \int_{-\infty}^x \left(y^{-1}\varphi(y)\right)' = \int_x^\infty (1 + y^{-2})\varphi(y) \le (1 + x^{-2})\mathbb{P}(X > x)$$
Therefore, $\mathbb{P}(X > x) \ge x^{-1}(1 + x^{-2})^{-1}\varphi(x) = (x + x^{-1})^{-1}\varphi(x)$.

Agenda

Comments and Reference

Obtaining exponential bounds is related to the very powerful theory of large deviations - see Varadhan (1984), Deuschel and Stroock (1989) - which has an ever-growing number of fields of application. See Ellis (1985).

You can study exponential bounds in the very specific context of martingales in Neveu (1975), Chow and Teicher (1978), Garsia (1973), etc.

Much of the literature is concerned with obtaining exponential bounds which are in a sense best possible. However, 'elementary' results such as the Azuma-HoefFding inequality in Exercise E14.1 are very useful in numerous applications. See for example the applications to combinatorics in Bollobas (1987).

Agenda

A consequence of Holder's inequality

Theorem

Suppose X and Y are non-negative R.V.s such that

$$c\mathbb{P}(X \geq c) \leq \mathbb{E}(Y; X \geq c),\ \forall c > 0$$

Then, for $p > 1$ and $p^{-1} + q^{-1} = 1$, we have $\|X\|_p \leq q\|Y\|_p$

Proof

Notice that by assumption, $L \triangleq \int_0^\infty pc^{p-1}\mathbb{P}(X \geq c)dc \leq \int_0^\infty pc^{p-2}\mathbb{E}(Y; X \geq c)dc \triangleq R$.
Using Fubini's Theorem with non-negative integrands, we obtain

$$L = \int_0^\infty \left(\int_\Omega \mathbb{1}(X \geq c)(\omega)\mathbb{P}(d\omega) \right) pc^{p-1}dc = \int_\Omega \left(\int_0^{X(\omega)} pc^{p-1}dc \right) \mathbb{P}(d\omega) = \mathbb{E}(X^p)$$

Similarly, we have

$$\begin{aligned} R &= \int_0^\infty pc^{p-2} \left(\int_\Omega Y(\omega)\mathbb{1}\left(X(\omega) \geq c\right)d\mathbb{P} \right) dc = \int_\Omega \int_0^{X(\omega)} Y(\omega)pc^{p-2}dcd\mathbb{P} \\ &= \frac{p}{p-1} \int Y(\omega)X(\omega)^{p-1}d\mathbb{P} = q\mathbb{E}(X^{p-1}Y). \end{aligned}$$

Proof (Cont)

By the inequality above and Holder inequality,

$$\mathbb{E}(X^p) \le \mathbb{E}(qX^{p-1}Y) \overset{(Holder)}{\le} q\|Y\|_p\|X^{p-1}\|_q$$

Suppose $\|Y\|_p < \infty$ (Otherwise trivial). Further suppose $\|X\|_p < \infty$. By noticing $(p-1)q = p$, $\|X^{p-1}\|_q = \mathbb{E}(X^p)^{1/q}$ and hence $\|X^p\| \le q\|Y\|_p$. Without assuming $\|X\|_p < \infty$, the result also holds by using $X \wedge n$ instead of X to the argument above follows by MON.

Agenda

Doob's $\mathcal{L}^p$ inequality

Theorem

1. Let $p > 1$ and define q so that $p^{-1} + q^{-1} = 1$. Let Z be non-negative submartingale bounded in $\mathcal{L}^p$. Define $Z^* \triangleq \sup_{k \in \mathbb{Z}+} Z_k$. Then $Z^* \in \mathcal{L}^p$ and indeed
$$\|Z^*\|_p \leq q \sup_r \|Z_r\|_p$$
, i.e. the submartingale is therefore dominated by the element Z^* in $\mathcal{L}^p$. Also, as $n \to \infty$, $Z_\infty \triangleq \lim_{n\to\infty} Z_n$ exists a.s. in $\mathcal{L}^p$ and
$$\|Z_\infty\|_p = \sup_r \|Z_r\|_p = \uparrow \lim_{r\to\infty} \|Z_r\|_p.$$
2. If Z is of the form $|M|$, where M is martingale bounded in $\mathcal{L}^p$, then $M_\infty = \lim_{n\to\infty} M_n$ exits a.s. and in $\mathcal{L}^p$. And of course $Z_\infty = |M_\infty|$ a.s.

Proof

For $n \in \mathbb{Z}^+$, define $Z_n^* \triangleq \sup_{k \leq n} Z_k$. By Doob's Submartingale inequality,

$$c\mathbb{P}(Z_n^* \geq c) \leq \mathbb{E}(Z_n; Z_N^* \geq c), \ \forall c > 0$$

, which satisfy the assumption in 14.10. Hence we have

$$\|Z_n^*\|_p \leq q\|Z_n\|_p \leq q \sup_r \|Z_r\|_p$$

Then by MON, $\|Z_n^*\|_p \leq q \sup_r \|Z_r\|_p$. Notice that $-Z$ is a Supermartingale in $\mathscr{L}^p$ and hence also in $\mathscr{L}^1$ (By Monotonicity of norms). By Doob's Forward Convergence Theorem, $Z_\infty \triangleq \lim_{n\to\infty} Z_n$ exist a.s. It suffices to show that Z_∞ exists in $\mathscr{L}^p$ sense. Notice that

$$|Z_n - Z_\infty|^p \leq (2Z^*)^p \in \mathscr{L}^p$$

Therefore, we can apply Dominated convergence theorem and $Z_n \xrightarrow{\mathscr{L}^p} Z_\infty$. By Jensen's inequality, as $\|Z_r\|_p$ is a non-decreasing sequence of r in $\mathbb{R}$. The result follows.

Agenda

Kakutani's theorem on 'product' martingales

Theorem

Let $X_1, X_2, \cdots$ be independent non-negative R.V.s, each of mean 1. Define $M_0 \triangleq 1$, and for $n \in \mathbb{N}$, let

$$M_n \triangleq X_1 X_2 \cdots X_n.$$

Then M is a non-negative martingale, so that

$$M_\infty \triangleq \lim_{n\to\infty} M_n \quad \text{exists a.s.}$$

And the following five statements are equivalent.

1. $\mathbb{E}(M_\infty) = 1$.
2. $M_n \to M_\infty$ in $\mathscr{L}^1$.
3. M is U.I.
4. $\prod a_n > 0$, where $0 < a_n \triangleq \mathbb{E}(X_n^{1/2}) \leq 1$.
5. $\sum(1 - a_n) < \infty$.

If one of the above statements fails to hold, then $\mathbb{P}(M_\infty = 0) = 1$.

Kakutani's theorem on 'product' martingales

Proof

As $x \mapsto x^{1/2}$ is a concave (NOT convex) function. By Jensen's inequality (converse), we have

$$0 < a_n \triangleq \mathbb{E}(X_n^{1/2}) \leq \sqrt{\mathbb{E}(X_n)} = 1$$

Suppose (4) holds. Define the sequence of R.V.s N_n by

$$N_n \triangleq \frac{X_1^{1/2}}{a_1} \times \frac{X_2^{1/2}}{a_2} \times \cdots \times \frac{X_n^{1/2}}{a_n}$$

Then $\mathbb{E}(N_{n+1}|\mathscr{F}_n) = \mathbb{E}(N_n X_{n+1}^{1/2}/a_n|\mathscr{F}_n) = N_n$, i.e. N is a martingale. Notice that

$$\mathbb{E}N_n^2 = 1/(a_1 a_1 \cdots a_n)^2 \leq 1/(\prod a_k)^2 < \infty$$

and hence N is bounded in $\mathscr{L}^2$. By Doob's $\mathscr{L}^2$ inequality, as $|M_n| \leq |N_n|^2$,

$$\mathbb{E}\left(\sup_n |M_n|\right) \leq \mathbb{E}\left(\sup_n |N_n|^2\right) \leq 2^2 \sup_r \mathbb{E}(|N_n^2|) < \infty$$

so that M is doinated by $M^* \triangleq \sup_n |M_n| \in \mathscr{L}^1$. Hence M is U.I. and (1)-(3) hold. The equiavlence of (4) and (5) follows from section (4.3). Suppose $\prod a_n = 0$, as N is non-negative martingale, $\lim_{n\to\infty} N_n$ exists a.s. Since $\prod a_n = 0$, we are forced to conclude $M_\infty = 0$ a.s.

Agenda

The σ-algebra $\mathscr{F}_T$, T is a stoping time

Definition of $\mathscr{F}_T$

Recall that a map $T : \Omega \mapsto \mathbb{Z}^+ \cup \{\infty\}$ is called a stopping time if $\{T = n\} \in \mathscr{F}_n, \ \forall n \in \mathbb{Z}^+ \cup \{\infty\}$. We then extend the definition to define $\mathscr{F}_T$. Let T be a stopping time. Then, for $F \subseteq \Omega$, we say that $F \in \mathscr{F}_T$ if

$$F \cap \{T = n\} \in \mathscr{F}_n, \ \forall n \in \mathbb{Z}^+ \cup \{\infty\}$$

Then $\mathscr{F}_T$ is also a σ-algebra.

Proof

1. Obviously, $\emptyset \cap \{T = n\} = \emptyset \in \mathscr{F}_n, \ \forall n \in \mathbb{Z}^+ \cup \{\infty\}$. Hence $\emptyset \in \mathscr{F}_T$.
2. Suppose $F \in \mathscr{F}_T$, then $F \cap \{T = n\} \in \mathscr{F}_n, \forall n \in \mathbb{Z}^+ \cup \{\infty\}$. It follows by the property of $\mathscr{F}_n$ being an σ-algebra that $F^c \cup \{T \neq n\} \in \mathscr{F}_n$ and also as T being stopping time implies $\{T = n\} \in \mathscr{F}_n$,
$$F^c \cup \{T = n\} = (F^c \cup \{T \neq n\}) \cup \{T = n\} \in \mathscr{F}_n$$
and hence $F^c \in \mathscr{F}_T$.
3. Suppose $F_1, F_2, \cdots \in \mathscr{F}_T$, then
$$\left\{F_i \cap \{T = n\}\right\}_{i=1}^{\infty} \in \mathscr{F}_n \Rightarrow (\cup_n F_i) \cap \{T = n\} = \cap_n \left(F_i \cap \{T = n\}\right) \in \mathscr{F}_n.$$
Hence $\cup_n F_i \in \mathscr{F}_T$ and $\mathscr{F}_T$ is a σ-algebra.

A special case of Optional-Sampling Theorem

Theorem

Let X be a supermartingale. Let T be a stopping time such that, for some $N \in \mathbb{N}$, $T(\omega) \leq N$, $\forall \omega$. Then $X_T \in \mathscr{L}^1(\Omega, \mathscr{F}_T, \mathbb{P})$ and

$$\mathbb{E}(X_N | \mathscr{F}_T) \leq X_T$$

Proof

Let $F \in \mathscr{F}_T$, then

$$\mathbb{E}(X_n; F) = \sum_{n \leq N} \mathbb{E}(X_N; F \cap \{T = n\}) \leq \sum_{n \leq N} \mathbb{E}(X_n; F \cap \{T = n\}) = \mathbb{E}(X_T; F)$$

Doob's OST for U.I. Martingales

Theorem

Let M be a UI Martingale. Then, for any stopping time T, $\mathbb{E}(M_\infty|\mathscr{F}_T) = M_T$

Proof

By Theorem 14.1 and Theorem in previous page, we have for $k \in \mathbb{N}$,

$$\mathbb{E}(M_\infty|\mathscr{F}_k) = M_k \quad \text{and} \quad \mathbb{E}(M_k|\mathscr{F}_{T\wedge k}) = M_{T\wedge k}, \quad a.s.$$

By Tower property, as $\mathscr{F}_{T\wedge k} \subseteq \mathscr{F}_k$, we have

$$\mathbb{E}(M_\infty|\mathscr{F}_{T\wedge k}) = \mathbb{E}\Big\{\mathbb{E}(M_\infty|\mathscr{F}_k)\Big|\mathscr{F}_{T\wedge k}\Big\} = \mathbb{E}(M_k|\mathscr{F}_{T\wedge k}) = M_{T\wedge k} \text{ a.s.}$$

If $F \in \mathscr{F}_T$, then $F \cap \{T \leq k\} \in \mathscr{F}_{T\wedge k}$, so that

$$\mathbb{E}(M_\infty; F \cap \{T \leq k\}) = \mathbb{E}(M_{T\wedge k}; F \cap \{T \leq k\}) = \mathbb{E}(M_T; F \cap \{T \leq k\})$$

WLOG, assume $M_\infty \geq 0$ and hence $M_n = \mathbb{E}(M_\infty|\mathscr{F}_n) \geq 0$ by positivity of conditional expectation. By letting $k \uparrow \infty$ and MON, we obtained

$$\mathbb{E}(M_\infty; F \cap \{T < \infty\}) = \mathbb{E}(M_T; F \cap \{T \leq < \infty\})$$

The equality $\mathbb{E}(M_\infty; F \cap \{T = \infty\}) = \mathbb{E}(M_T; F \cap \{T \leq = \infty\})$ is trivial. It follows that $\mathbb{E}(M_\infty; F) = \mathbb{E}(M_T; F)$ for all $F \in \mathscr{F}_T$. The proof is completed.

Doob's OST for U.I. Martingales

Corollary

If M is a UI Martingale and S and T are both stopping time with $S \leq T$, then

$$\mathbb{E}(M_T|\mathscr{F}_S) = M_S, \text{ a.s.}$$

Proof of Corollary

Recall from previous theorem, $\mathbb{E}(M_\infty|\mathscr{F}_T) = M_T$ a.s. Hence we have

$$\mathbb{E}(M_T|\mathscr{F}_S) = \mathbb{E}\Big\{\mathbb{E}(M_\infty|\mathscr{F}_T)\Big|\mathscr{F}_S\Big\} \overset{(\mathscr{F}_S \subseteq \mathscr{F}_T)}{=} \mathbb{E}(M_\infty|\mathscr{F}_S) = M_S \text{ a.s.}$$

A new Optional-Stopping Theorem

If M is a UI Martingale, and T is a stopping time, then $\mathbb{E}(|M_T|) < \infty$ and

$$\mathbb{E}(M_T) = \mathbb{E}(M_0)$$

, which is a direct consequence of corollary above.